Chapter 1. Probability Theory
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CHAPTER 1 Probability Theory
1.1 Probabilities
1.1.1 Introduction

- Statistics and Probability theory constitutes a branch of mathematics for dealing with **uncertainty**

- Probability theory provides a basis for the science of **statistical inference** from data

CHAPTER 1 Probability Theory
1.1 Probabilities
1.1.2 Sample Spaces(1/3)

- **Experiment** : any process or procedure for which more than one **outcome** is possible

- **Sample Space**

  The **sample space** S of an experiment is a set consisting of all of the possible experimental outcomes.

1.1.2 Sample Spaces(2/3)

- Example 3: **Software Errors**

  The number of separate errors in a particular piece of software can be viewed as having a **sample space**

$$S = \{0 \text{ errors}, 1 \text{errors}, 2 \text{errors}, 3 \text{errors}, \ldots\}$$

- Example 4: **Power Plant Operation**

  A manager supervises the operation of three power plants, at any given time, each of the three plants can be classified as either generating electricity (1) or being idle (0).

$$S = \{(0,0,0)\ (0,0,1)\ (0,1,0)\ (0,1,1)\ (1,0,0)\ (1,0,1)\ (1,1,0)\ (1,1,1)\}$$

1.1.2 Sample Spaces(3/3)

- GAMES OF CHANCE
  - Games of chance commonly involve the toss of a coin, the roll of a die, or the use of a pack of cards.
  - <u>The roll of a die</u>

    A usual six-sided die has a sample space

$$S = \{1, 2, 3, 4, 5, 6\}$$

    If two dice are rolled (or, equivalently, if one die is rolled twice), the sample space is shown in Figure 1.2.

FIGURE 1.2 ● Sample space for rolling two dice

S

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

1.1.3 Probability Values(1/5)

- **Probabilities**

A set of **probability values** for an experiment with a sample space $S = \{O_1, O_2, \cdots, O_n\}$ consists of some probabilities

$p_1, p_2, \cdots, p_n$

that satisfy

$$0 \le p_1 \le 1, 0 \le p_2 \le 1, \cdots, 0 \le p_n \le 1$$

and

$$p_1 + p_2 + \cdots + p_n = 1$$

The probability of outcome O_i occurring is said to be p_i, and this is written

$$P(O_i) = p_i$$

1.1.3 Probability Values(2/5)

- Example 3 **Software Errors**

  Suppose that the numbers of errors in a software product have probabilities

$$P(0 \text{ errors}) = 0.05,\ P(1 \text{ error}) = 0.08,\ P(2 \text{ errors}) = 0.35,$$

$$P(3 \text{errors}) = 0.20,\ P(4 \text{ errors}) = 0.20,\ P(5 \text{ errors}) = 0.12,$$

$$P(i \text{ errors}) = 0, \quad \text{for } i \geq 6$$

  There are at most 5 errors since the probability values are zero for 6 or more errors.

  The most likely number of errors is 2.

  3 and 4 errors are equally likely in the software product.

1.1.3 Probability Values(3/5)

- In some situations, notably games of chance, the experiments are conducted in such a way that all of the possible outcomes can be considered to be **equally likely**, so that they must be assigned **identical probability values**.

- **n** outcomes in the sample space that are equally likely => each probability value be **1/n**.

1.1.3 Probability Values(4/5)

- GAMES OF CHANCE
  - A fair die will have each of the six outcomes equally likely.

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

  - An example of a biased die would be one of which

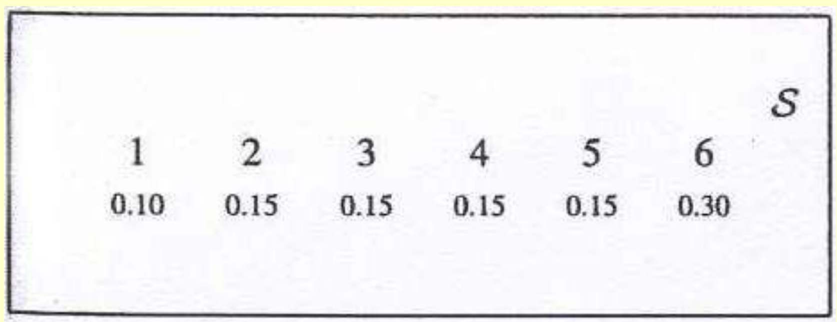


FIGURE 1.9 ●
Probability values for a biased die

In this case the die is most likely to score a 6, which will happen **roughly** three times out of ten as a long-run average.

1.1.3 Probability Values(5/5)

- If two die are thrown and each of the 36 outcomes is **equally likely** (as will be the case two fair dice that are shaken properly), the probability value of each outcome will necessarily be 1/36

S

(1, 1) 1/36	(1, 2) 1/36	(1, 3) 1/36	(1, 4) 1/36	(1, 5) 1/36	(1, 6) 1/36
(2, 1) 1/36	(2, 2) 1/36	(2, 3) 1/36	(2, 4) 1/36	(2, 5) 1/36	(2, 6) 1/36
(3, 1) 1/36	(3, 2) 1/36	(3, 3) 1/36	(3, 4) 1/36	(3, 5) 1/36	(3, 6) 1/36
(4, 1) 1/36	(4, 2) 1/36	(4, 3) 1/36	(4, 4) 1/36	(4, 5) 1/36	(4, 6) 1/36
(5, 1) 1/36	(5, 2) 1/36	(5, 3) 1/36	(5, 4) 1/36	(5, 5) 1/36	(5, 6) 1/36
(6, 1) 1/36	(6, 2) 1/36	(6, 3) 1/36	(6, 4) 1/36	(6, 5) 1/36	(6, 6) 1/36

FIGURE 1.10 ●
Probability values for rolling two dice

1.2 Events
1.2.1 Events and Complements(1/6)

- Events

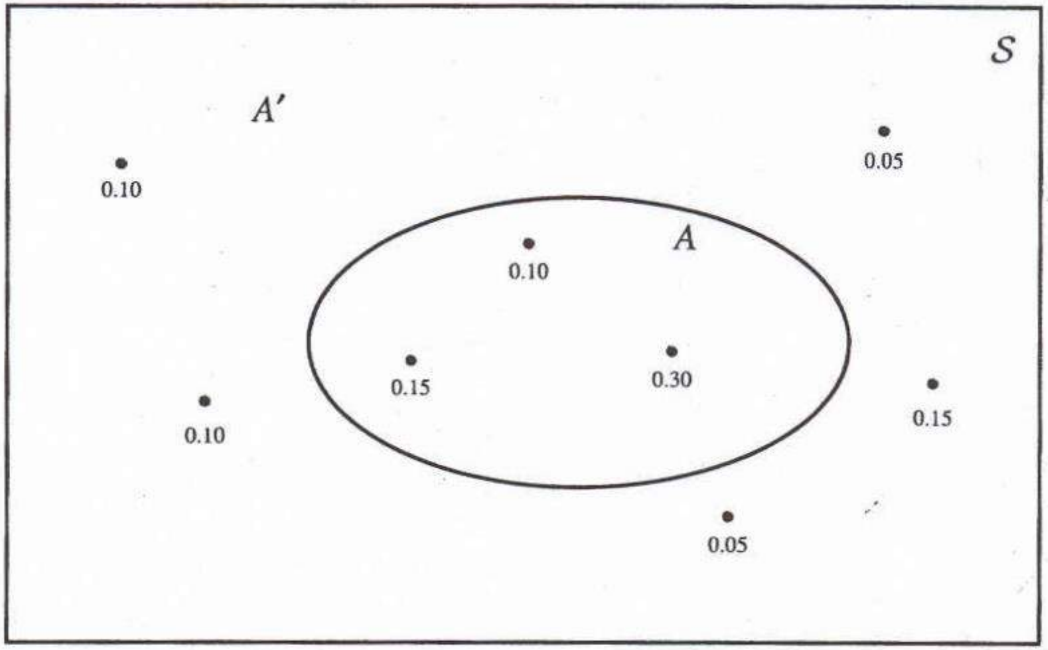
  An **event** *A* is a subset of the sample space *S*. It collects outcomes of particular interest. The probability of an event A, $P(A)$, is obtained by summing the probabilities of the outcomes contained within the event *A*.

- An **event** is said to **occur** if one of the outcomes contained within the event occurs.

1.2.1 Events and Complements(2/6)

- A sample space S consists of eight outcomes with a probability value.

FIGURE 1.14 ●
$P(A) = 0.10 + 0.15 + 0.30 = 0.55$



$$P(A) = 0.10 + 0.15 + 0.30 = 0.55$$

$$P(A') = 0.10 + 0.05 + 0.05 + 0.15 + 0.10 = 0.45$$

Notice that $P(A) + P(A') = 1.$

1.2.1 Events and Complements(3/6)

- **Complements of Events**

  The event A' , the **complement** of event *A*, is the event consisting of everything in the sample space *S* that is not contained within the event *A*. In all cases

$$P(A) + P(A') = 1$$

- Events that consist of an individual outcome are sometimes referred to as **elementary events** or **simple events**

1.2.1 Events and Complements(4/6)

- Example 3 **Software Errors**

  Consider the event *A* that there are no more than two errors in a software product.

  $$A = \{\text{ 0 errors, 1 error, 2 errors }\} \subset S$$

  and

  $$P(A) = P(\text{0 errors}) + P(\text{1 error}) + P(\text{2 errors})$$

  $$= 0.05 + 0.08 + 0.35 = 0.48$$

  $$P(A') = 1 - P(A) = 1 - 0.48 = 0.52$$

1.2.1 Events and Complements(5/6)

- GAMES OF CHANCE
  - even = { an *even* score is recorded on the roll of a die }

    = { 2,4,6 }

    For a fair die, $P(\text{even}) = P(2) + P(4) + P(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$
  - A = { the sum of the scores of two dice is equal to 6 }

    = { (1,5), (2,4), (3,3), (4,2), (5,1) }

$$P(A) = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{5}{36}$$

A sum of 6 will be obtained with two fair dice roughly 5 times out of 36 on average, that is, on about 14% of the throws.

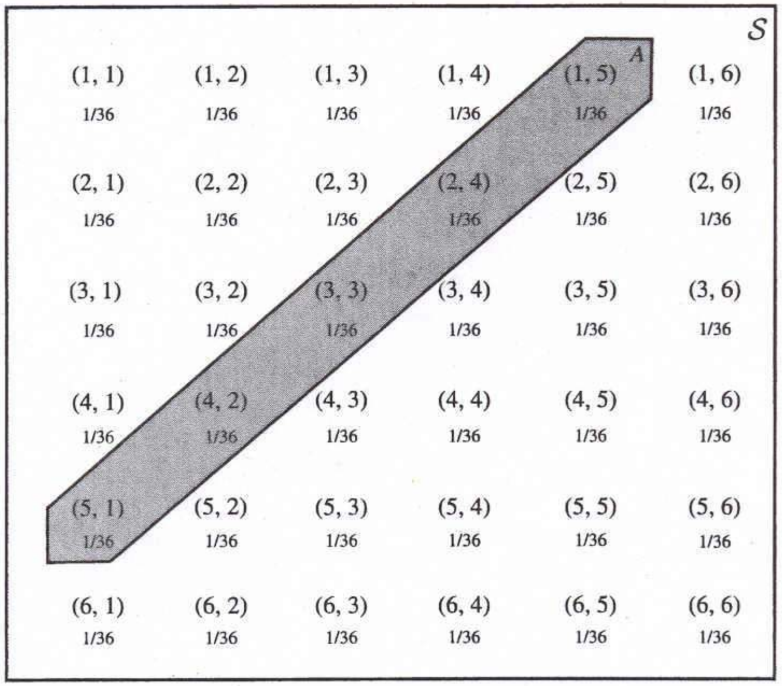


FIGURE 1.18 •
Event A: sum equal to 6

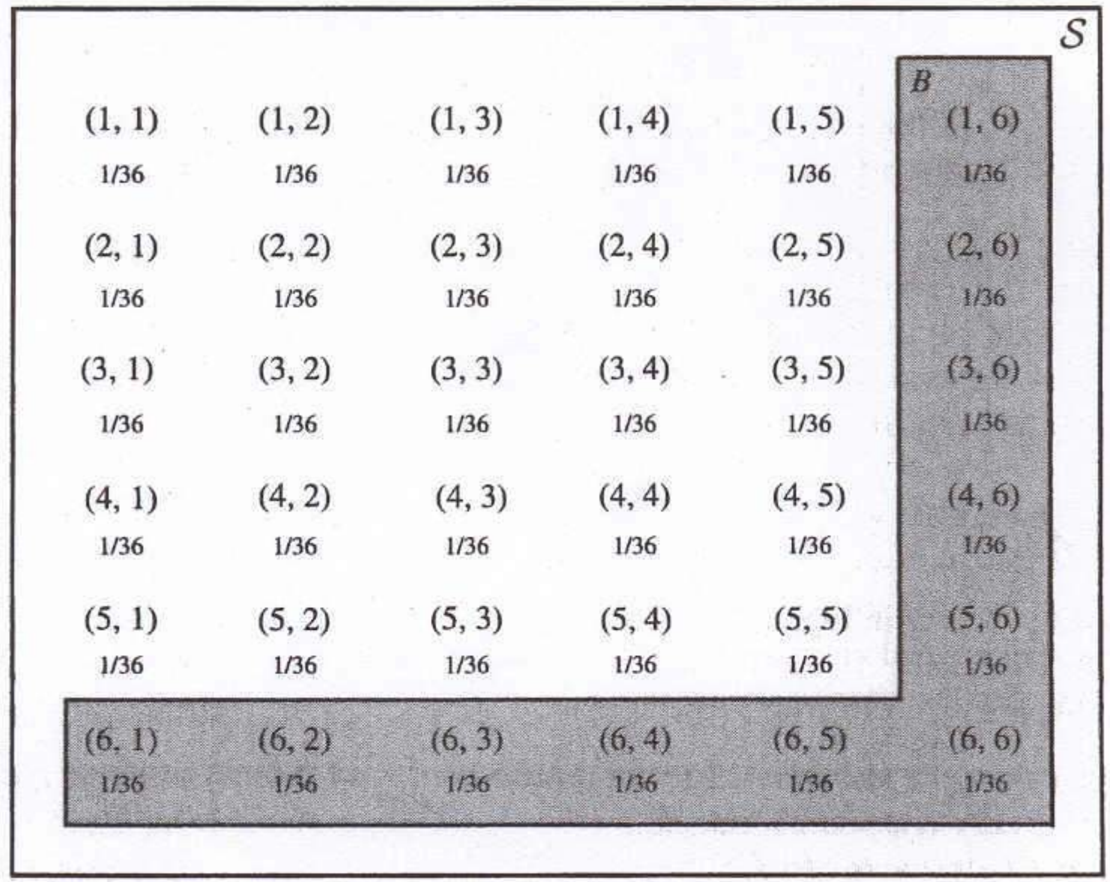
1.2.1 Events and Complements(6/6)

– B = { at least one of the two dice records a 6 }

FIGURE 1.19 ● Event B: at least one 6 recorded

$$P(B) = \frac{11}{36}$$

$$P(B') = 1 - \frac{11}{36} = \frac{25}{36}$$



1.3 Combinations of Events
1.3.1 Intersections of Events(1/5)

- Intersections of Events

The event $A \cap B$ is the **intersection** of the events *A* and *B* and consists of the outcomes that are contained within both events *A* and *B*. The probability of this event, $P(A \cap B)$, is the probability that both events *A* and *B* occur **simultaneously**.

1.3.1 Intersections of Events(2/5)

- A sample space S consists of 9 outcomes

FIGURE 1.26 Events A and B

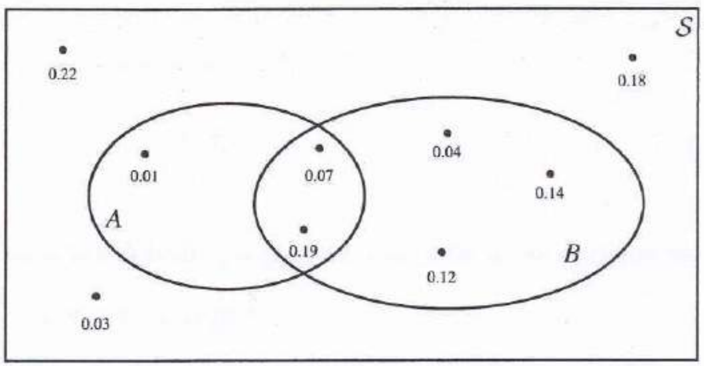
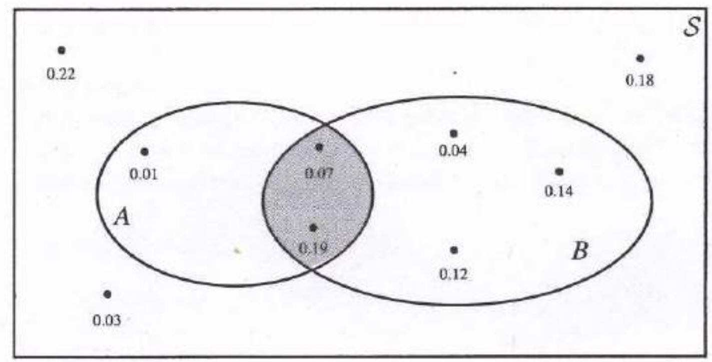


FIGURE 1.27 The event $A \cap B$



$$P(A) = 0.01 + 0.07 + 0.19 = 0.27$$

$$P(B) = 0.07 + 0.19 + 0.04 + 0.14 + 0.12 = 0.56$$

$$P(A \cap B) = 0.07 + 0.19 = 0.26$$

1.3.1 Intersections of Events(3/5)

FIGURE 1.28 ● The event $A' \cap B$

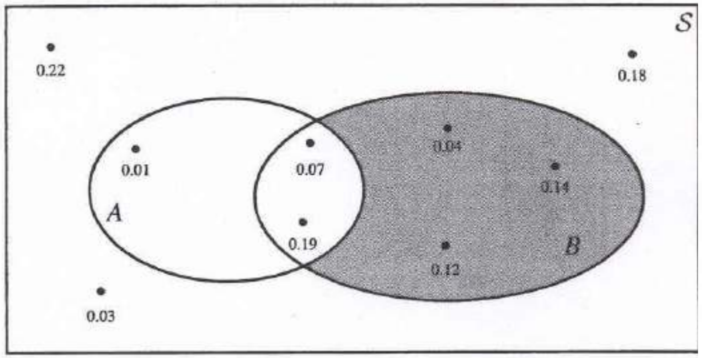
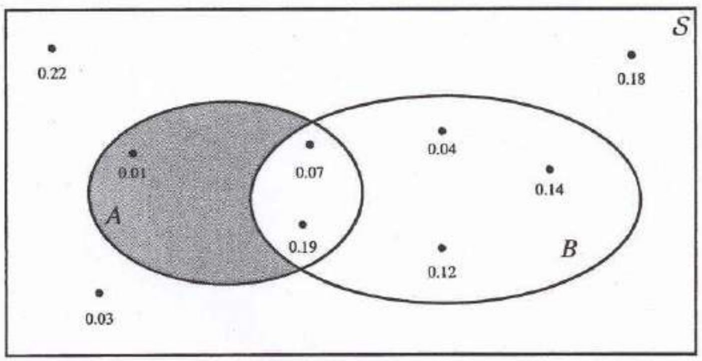


FIGURE 1.29 ● The event $A \cap B'$



$$P(A' \cap B) = 0.04 + 0.14 + 0.12 = 0.30$$

$$P(A \cap B') = 0.01$$

$$P(A \cap B) + P(A \cap B') = 0.26 + 0.01 = 0.27 = P(A)$$

$$P(A \cap B) + P(A' \cap B) = 0.26 + 0.30 = 0.56 = P(B)$$

1.3.1 Intersections of Events(4/5)

□ $P(A \cap B) + P(A \cap B') = P(A)$

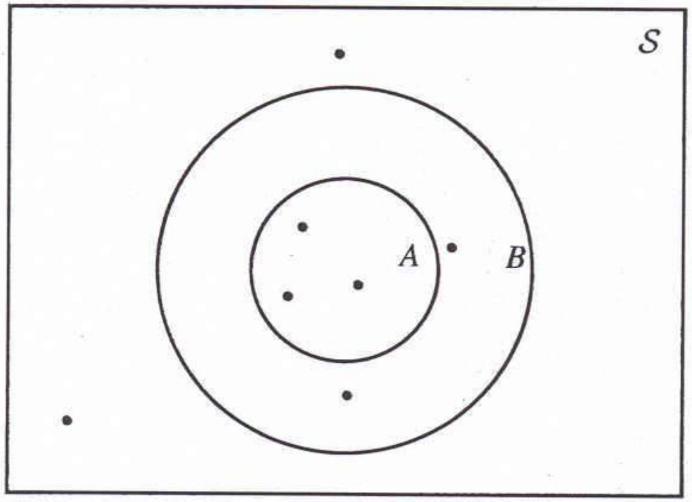
$P(A \cap B) + P(A' \cap B) = P(B)$

- Mutually Exclusive Events

  Two events *A* and *B* are said to be **mutually exclusive** if $A \cap B = \varnothing$ so that they have **no outcomes in common.**

1.3.1 Intersections of Events(5/5)

FIGURE 1.31 ●
$A \subset B$



$$A \cap B = A$$

- $A \cap B = B \cap A$

  $A \cap A = A$

  $A \cap S = A$

  $A \cap \varnothing = \varnothing$

  $A \cap A' = \varnothing$

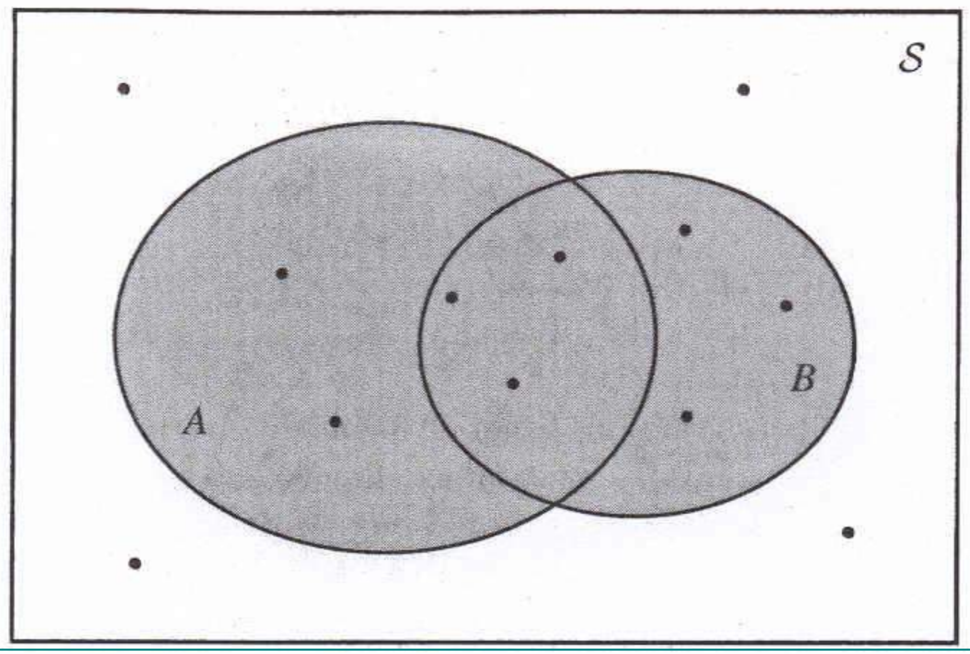
  $A \cap (B \cap C) = (A \cap B) \cap C$

1.3.2 Unions of Events(1/4)

- Unions of Events

The event $A \cup B$ is the **union** of events *A* and *B* and consists of the outcomes that are contained within at least one of the events *A* and *B*. The probability of this event, $P(A \cup B)$, is the probability that at least one of the events *A* and *B* occurs.

FIGURE 1.32
The event $A \cup B$

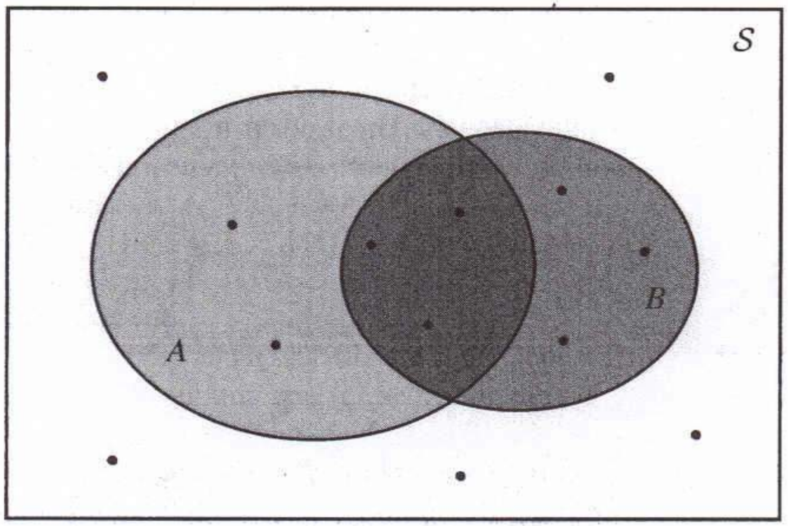


1.3.2 Unions of Events(2/4)

- Notice that the outcomes in the event $A \cup B$ can be classified into three kinds.

1. in event A but not in event B
2. in event B but not in event A or
3. in both events A and B

FIGURE 1.33 ●
Decomposition of the event $A \cup B$



$$P(A \cup B) = P(A \cap B') + P(A' \cap B) + P(A \cap B)$$

1.3.2 Unions of Events(3/4)

- $P(A \cap B') = P(A) - P(A \cap B)$

  $P(A' \cap B) = P(B) - P(A \cap B)$

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

- If the events A and B are mutually exclusive so that $P(A \cap B) = 0$, then $P(A \cup B) = P(A) + P(B)$

1.3.2 Unions of Events(4/4)

- Simple results concerning the unions of events

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

$$A \cup B = B \cup A$$

$$A \cup A = A$$

$$A \cup S = S$$

$$A \cup \emptyset = A$$

$$A \cup A' = S$$

$$A \cup (B \cup C) = (A \cup B) \cup C$$

1.3.3 Examples of Intersections and Unions(1/11)

- Example 5 **Television Set Quality**

A company that manufactures television sets performs a final quality check on each appliance before packing and shipping it. The quality check has an evaluation of the quality of the **picture** and the **appearance**. Each of the two evaluations is graded as *Perfect (P), Good (G), Satisfactory (S),* or *Fail (F).*

S

(P, P) 0.140	(P, G) 0.102	(P, S) 0.157	(P, F) 0.007
(G, P) 0.124	(G, G) 0.141	(G, S) 0.139	(G, F) 0.012
(S, P) 0.067	(S, G) 0.056	(S, S) 0.013	(S, F) 0.010
(F, P) 0.004	(F, G) 0.011	(F, S) 0.009	(F, F) 0.008

FIGURE 1.38 • Probability values for television set example

1.3.3 Examples of Intersections and Unions(2/11)

An appliance that fails on either of the two evaluations and that score an evaluation of *Satisfactory* on **both** accounts will **not be shipped**.

A = { an appliance cannot be shipped }

= { *(F,P), (F,G), (F,S), (F,F), (P,F), (G,F), (S,F), (S,S)* }

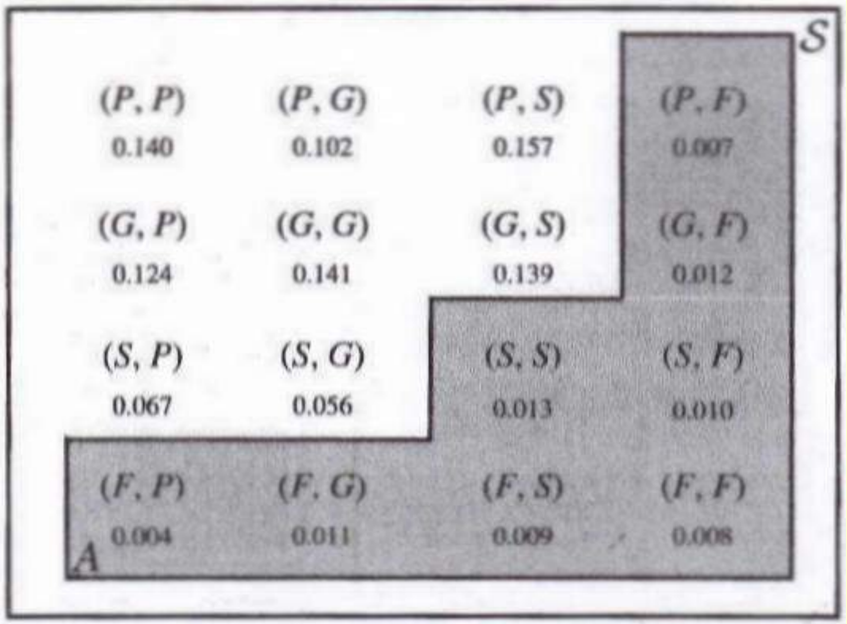


FIGURE 1.39 •
Event *A*: appliance not shipped

P(A) = 0.074

About 7.4% of the television sets will fail the quality check.

1.3.3 Examples of Intersections and Unions(3/11)

B = { picture satisfactory or fail }

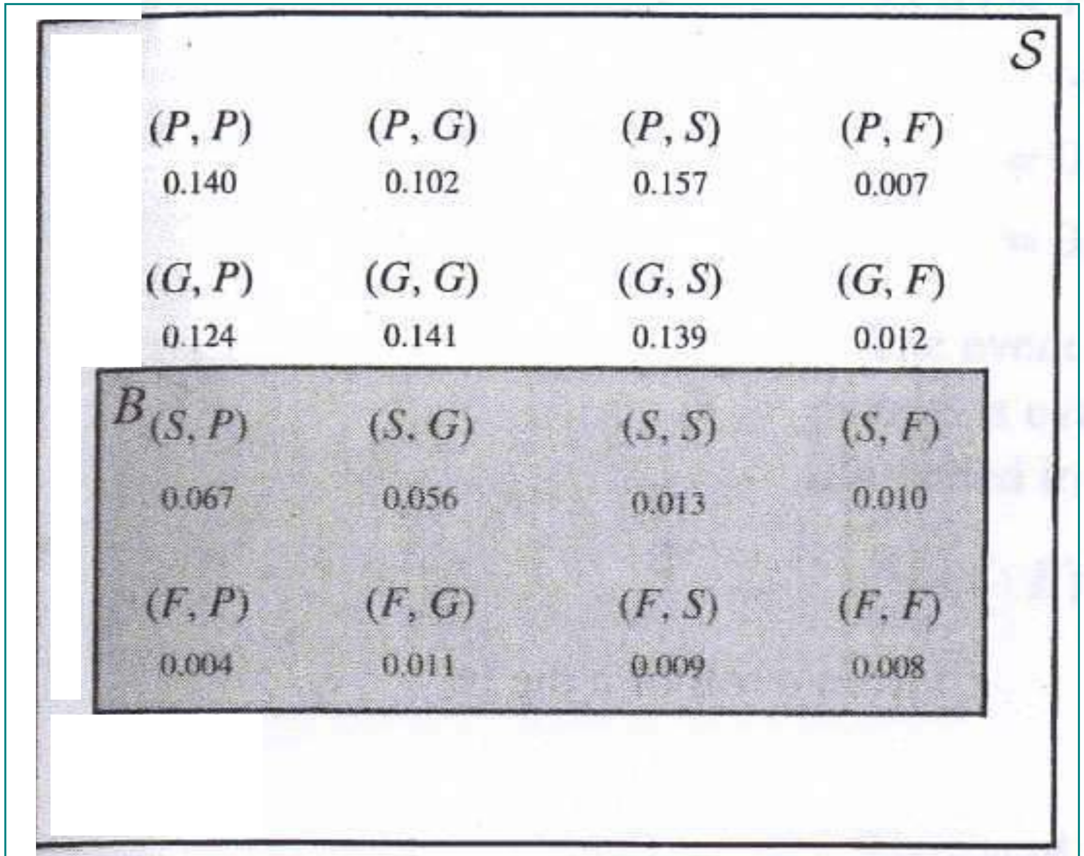


FIGURE 1.40 ●
Event *B*: picture satisfactory or fail

$P(B) = 0.178$

1.3.3 Examples of Intersections and Unions(4/11)

$A \cap B$ = { Not shipped **and** the picture satisfactory or fail }

$$P(A \cap B) = 0.055$$

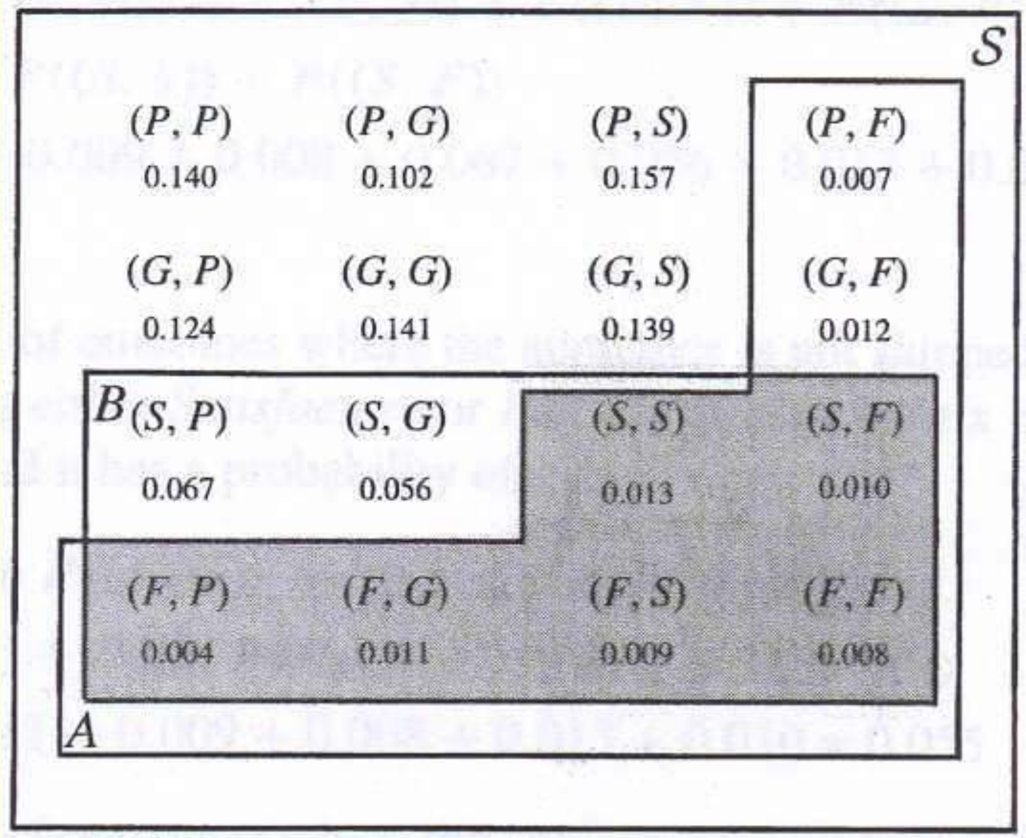


FIGURE 1.41 ●
Event $A \cap B$

1.3.3 Examples of Intersections and Unions(5/11)

$A \cup B$ = { the appliance was **either** not shipped **or** the picture was evaluated as being either *Satisfactory* or *Fail* }

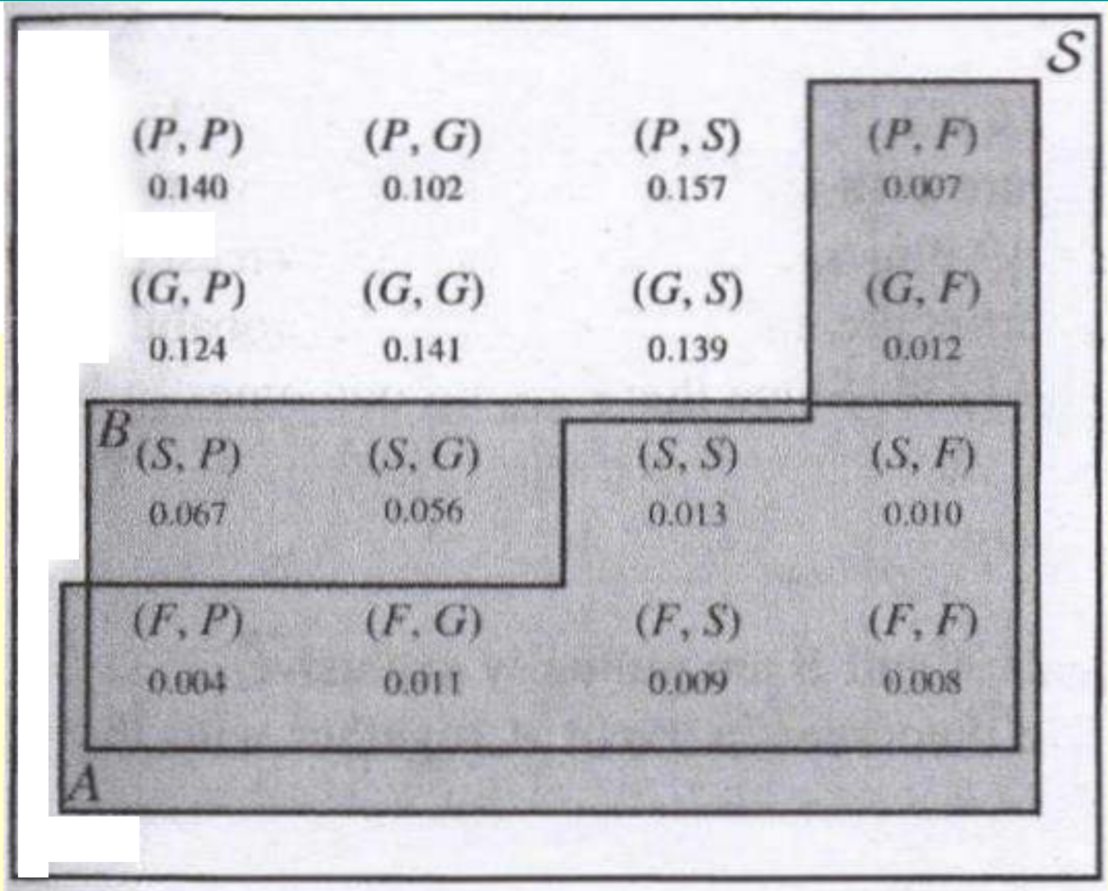


FIGURE 1.42 ●
Event $A \cup B$

$$P(A \cup B) = 0.197$$

1.3.3 Examples of Intersections and Unions(6/11)

$P(A \cap B')$ = { Television sets that have a picture evaluation of either *Perfect* or *Good* but that cannot be shipped }

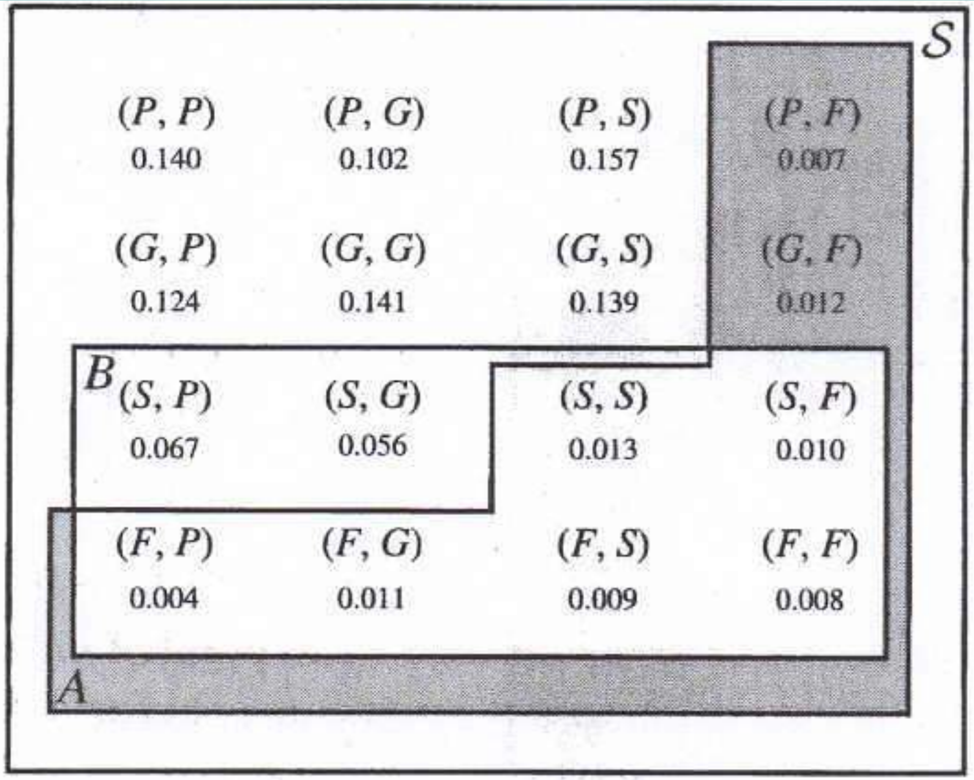


FIGURE 1.43 ● Event $A \cap B'$

$$P(A \cap B') = 0.019$$

Notice

$$P(A \cap B) + P(A \cap B') = 0.055 + 0.019$$
$$= 0.074$$
$$= P(A)$$

1.3.3 Examples of Intersections and Unions(7/11)

- GAMES OF CHANCE
  - A = { an even score is obtained from a roll of a die }
    = { 2, 4, 6 }

    B = { 4, 5, 6 }

    then $A \cap B = \{4, 6\}$ and $A \cup B = \{2, 4, 5, 6\}$

    $$P(A \cap B) = \frac{2}{6} = \frac{1}{3} \text{ and } P(A \cup B) = \frac{4}{6} = \frac{2}{3}$$
  - A = { the sum of the scores is equal to 6 }

    B = { at least one of the two dice records a 6 }

    $P(A) = 5/36$ and $P(B) = 11/36$

    $A \cap B = \emptyset$ and $P(A \cap B) = 0$

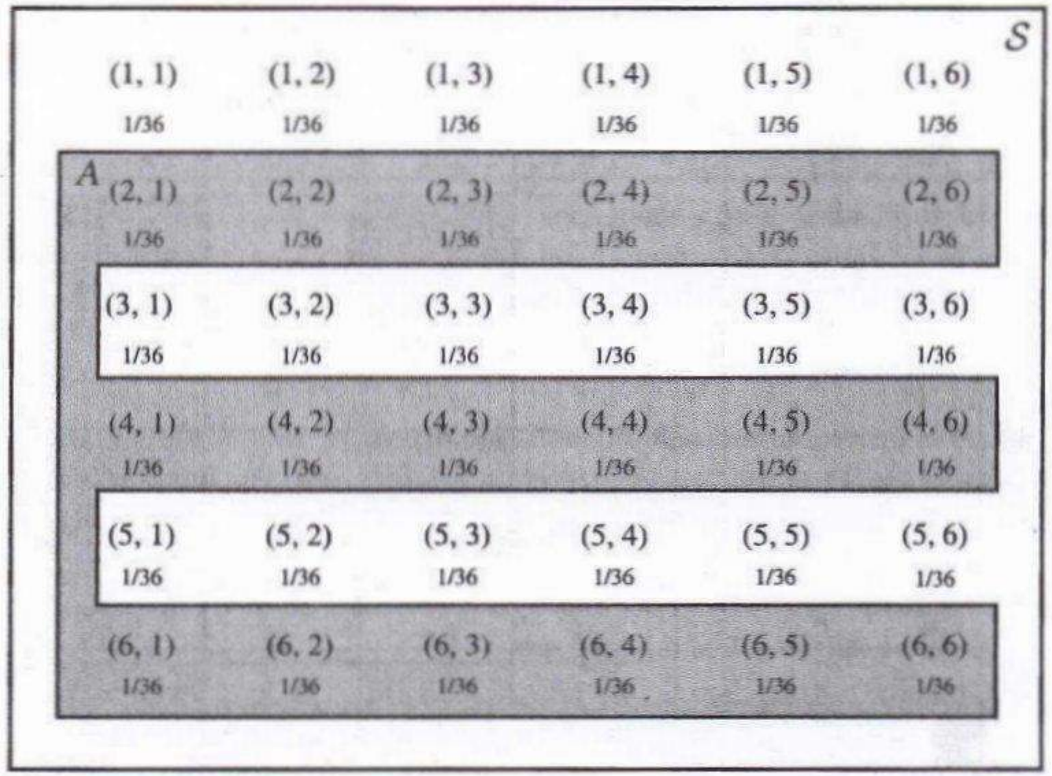
    => A and B are mutually exclusive

    $$P(A \cup B) = \frac{16}{36} = \frac{4}{9} = P(A) + P(B)$$

1.3.3 Examples of Intersections and Unions(8/11)

– One die is red and the other is blue ⟶ (red, blue).
  A = { an even score is obtained on the red die }

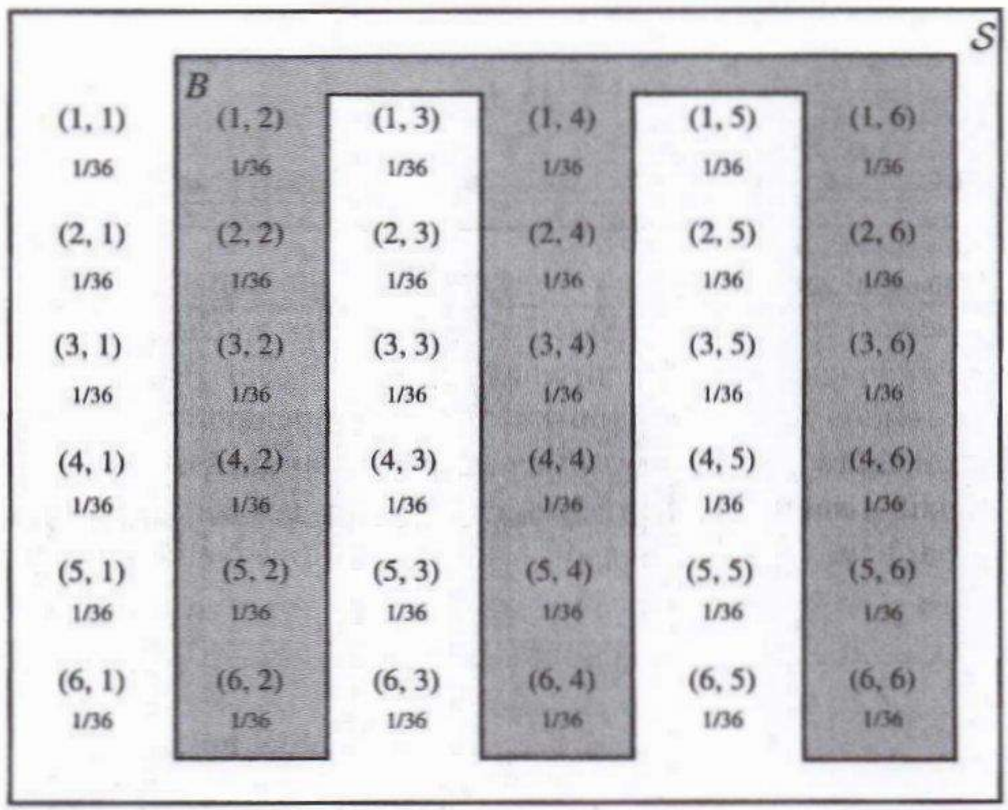
FIGURE 1.44 •
Event A: even score on red die



1.3.3 Examples of Intersections and Unions(9/11)

B = { an even score is obtained on the blue die }

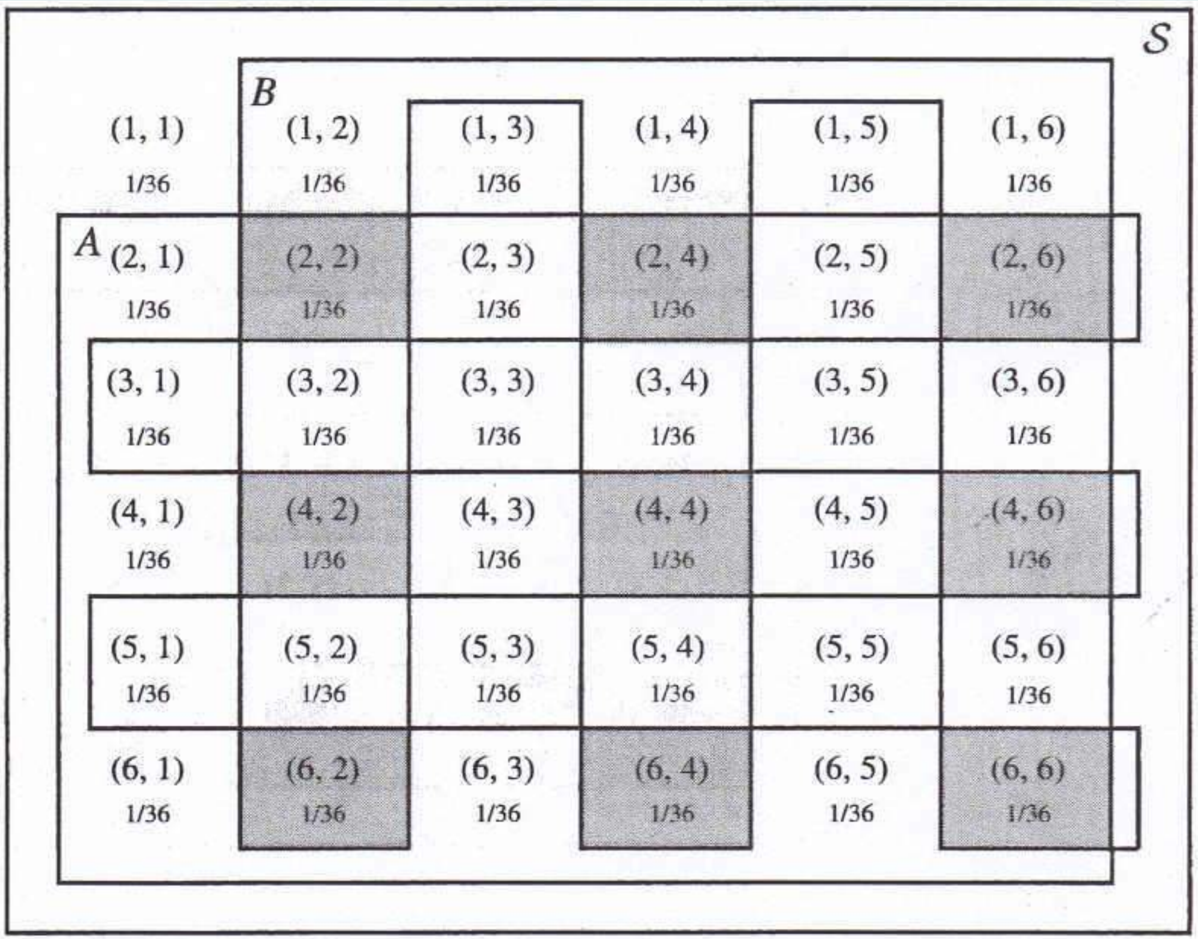
FIGURE 1.45 •
Event *B*: even score on blue die



1.3.3 Examples of Intersections and Unions(10/11)

$A \cap B$ = { both dice have even scores }

FIGURE 1.46 ● Event $A \cap B$

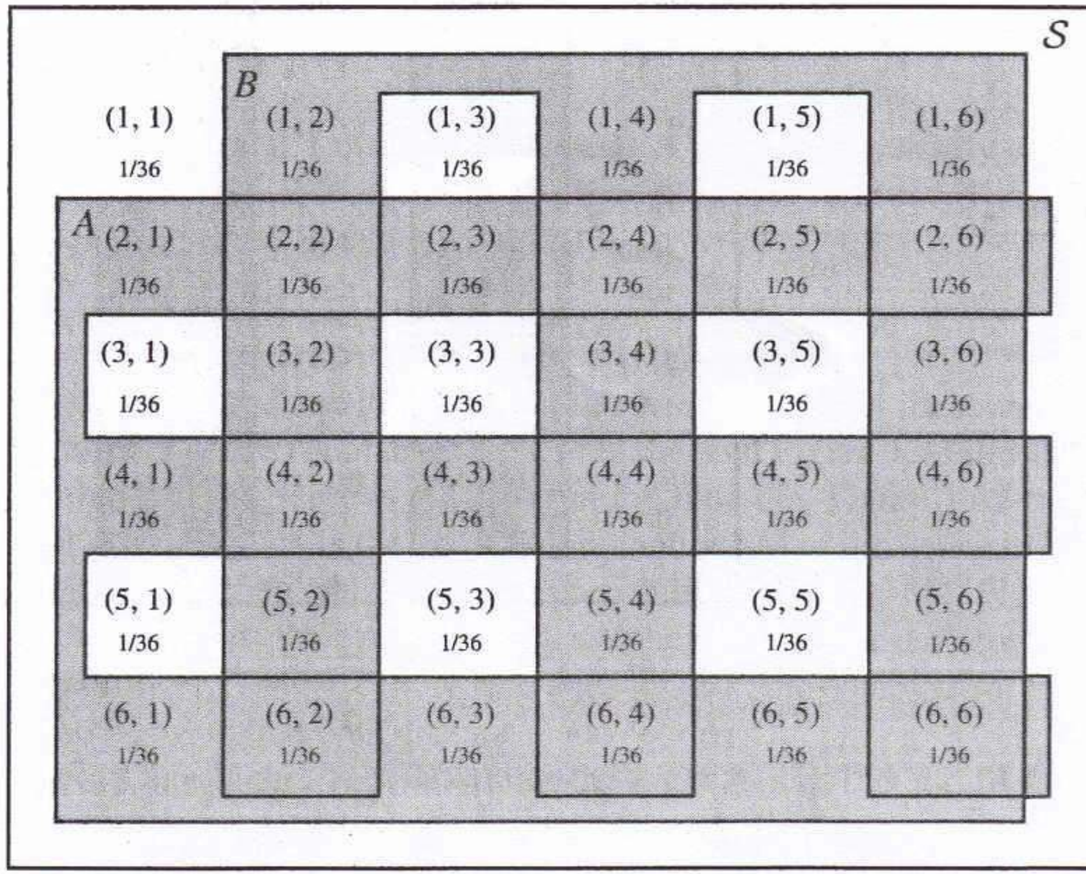


$$P(A \cap B) = \frac{9}{36} = \frac{1}{4}$$

1.3.3 Examples of Intersections and Unions(11/11)

$A \cup B$ = { at least one die has an even score }

FIGURE 1.47 ● Event $A \cup B$



$$P(A \cup B) = \frac{27}{36} = \frac{3}{4}$$

1.3.4 Combinations of Three or More Events(1/4)

- Union of Three Events

The probability of the **union of three events** *A, B,* and *C* is the sum of the probability values of the simple outcomes that are contained within at least one of the three events. It can also be calculated from the expression

$$P(A \cup B \cup C) = \left[P(A) + P(B) + P(C)\right] - \left[P(A \cap B) + P(A \cap C) + P(B \cap C)\right] + P(A \cap B \cap C)$$

1.3.4 Combinations of Three or More Events(2/4)

FIGURE 1.51 •
Three events decompose sample space into eight regions

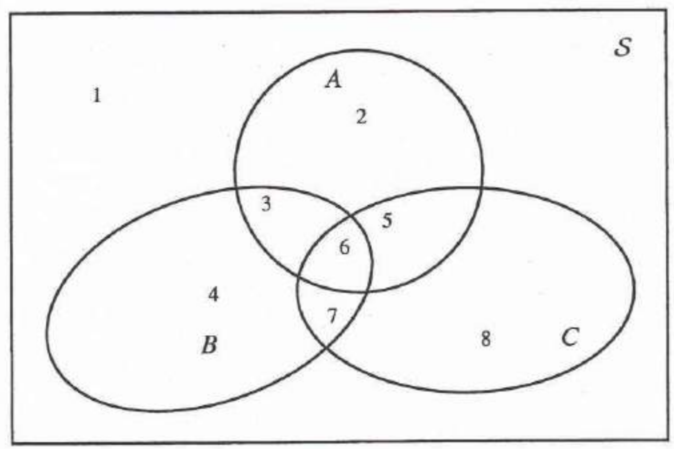
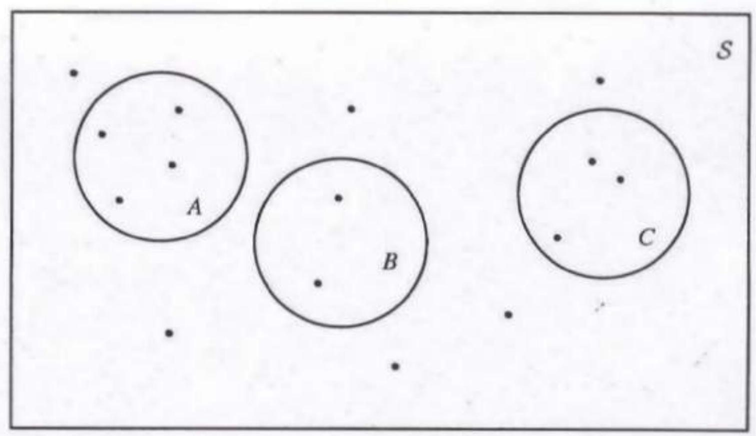


FIGURE 1.52 •
Three mutually exclusive events



$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

1.3.4 Combinations of Three or More Events(3/4)

- **Union of Mutually Exclusive Events**

For a sequence $A_1, A_2, ..., A_n$ of **mutually exclusive events**, the probability of the **union** of the events is given by

$$P(A_1 \cup \cdots \cup A_n) = P(A_1) + \cdots + P(A_n)$$

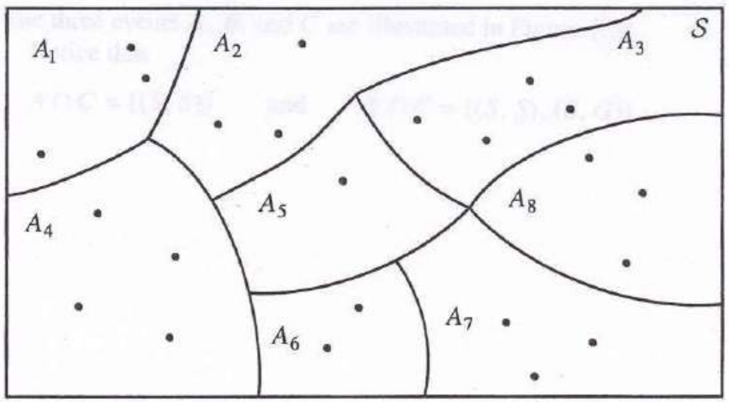
- **Sample Space Partitions**

A **partition** of a sample space is a sequence $A_1, A_2, ..., A_n$ of *mutually exclusive* events for which

$$A_1 \cup \cdots \cup A_n = S$$

Each outcome in the sample space is then contained within one and only one of the events A_i

FIGURE 1.53 •
A partition of the sample space



1.3.4 Combinations of Three or More Events(4/4)

- Example 5 **Television Set Quality**

  C = { an appliance is of "mediocre quality" }

  = { score either *Satisfactory* or *Good* }

  = { *(S,S), (S,G), (G,S), (G,G)* }

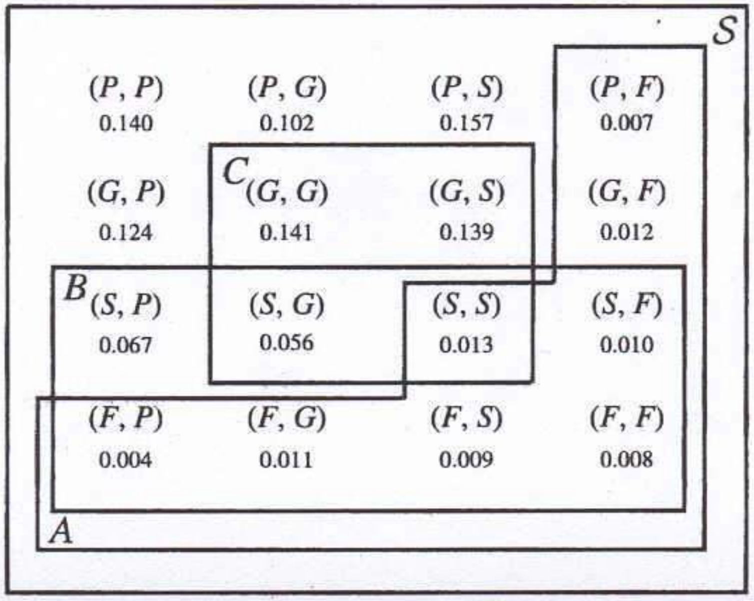
  D = { an appliance is of " high quality" }

  = $(A \cup B \cup C)' = A' \cap B' \cap C'$

  = { *(G,P), (P,P), (P,G), (P,S)* }

  P(D) = 0.523

FIGURE 1.54 ●
Events *A*, *B*, and *C*



1.4 Conditional Probability
1.4.1 Definition of Conditional Probability(1/2)

- **Conditional Probability**

  The **conditional probability** of event *A* conditional on event *B* is

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$

  for *P(B)>0*. It measures the probability that event *A* occurs when it is known that event *B* occurs.

□ $A\cap B=\varnothing$

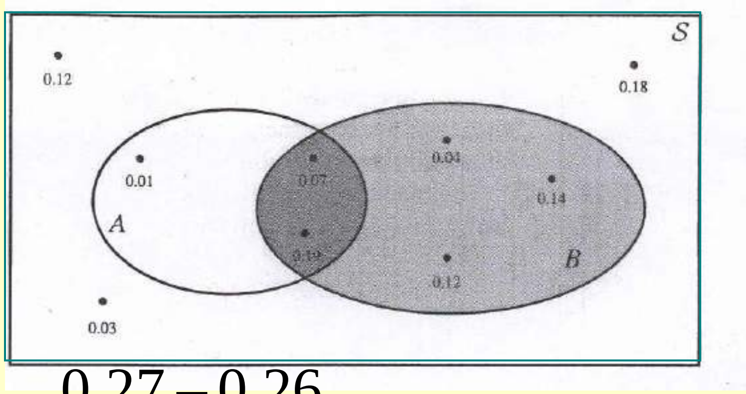
$$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{0}{P(B)}=0$$

□ $B\subset A$

$$A\cap B=B$$

$$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{P(B)}{P(B)}=1$$

1.4.1 Definition of Conditional Probability(2/2)



- $$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{0.26}{0.56}=0.464$$

$$P(A|B')=\frac{P(A\cap B')}{P(B')}=\frac{P(A)-P(A\cap B)}{1-P(B)}=\frac{0.27-0.26}{1-0.56}=0.023$$

$$\Rightarrow P(A|B)+P(A'|B)=1$$

Formally,

$$P(A|B)+P(A'|B)=\frac{P(A\cap B)}{P(B)}+\frac{P(A'\cap B)}{P(B)}$$

$$=\frac{1}{P(B)}(P(A\cap B)+P(A'\cap B))=\frac{1}{P(B)}P(B)=1$$

- $$P(A|B\cup C)=\frac{P(A\cap(B\cup C))}{P(B\cup C)}$$

1.4.2 Examples of Conditional Probabilities(1/3)

- Example 4 **Power Plant Operation**

  A = { plant X is idle } and <u>*P(A) = 0.32*</u>

  Suppose it is known that at least two out of the three plants are generating electricity (event B).

  B = { at least two out of the three plants generating electricity }

$$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{0.18}{0.70}=0.257$$

<u>*P(A) = 0.32*</u>

=> <u>*P(A/B) = 0.257*</u>

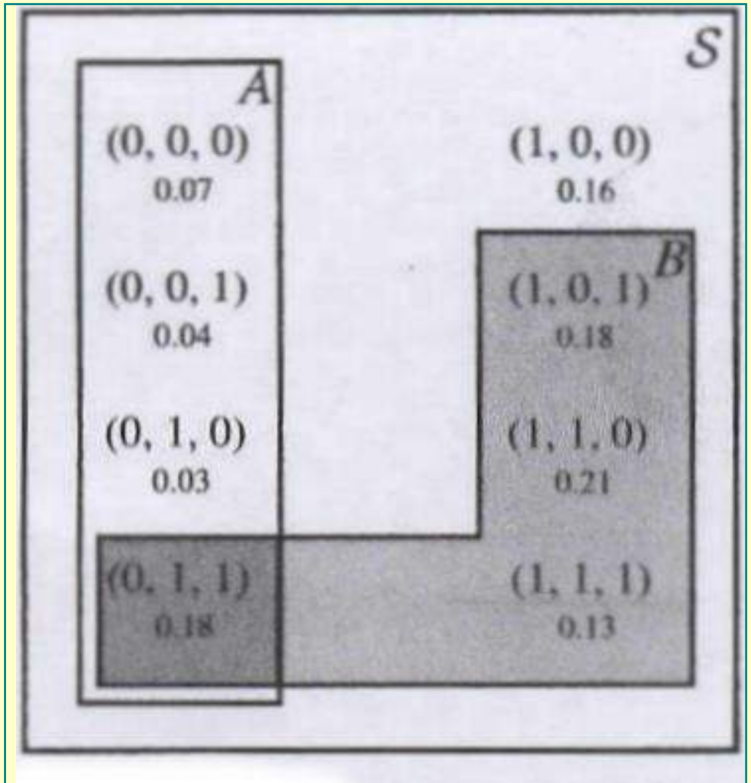


FIGURE 1.58 • $P(A|B) = P(A\cap B)/P(B)$

1.4.2 Examples of Conditional Probabilities(2/3)

- GAMES OF CHANCE
  - A fair die is rolled. $P(6)=\frac{1}{6}$

$$P(6\,|\,even)=\frac{P(6\cap even)}{P(even)}=\frac{P(6)}{P(even)}$$

$$=\frac{P(6)}{P(2)+P(4)+P(6)}=\frac{1/6}{1/6+1/6+1/6}=\frac{1}{3}$$

  - A red die and a blue die are thrown.

    A = { the red die scores a 6 }

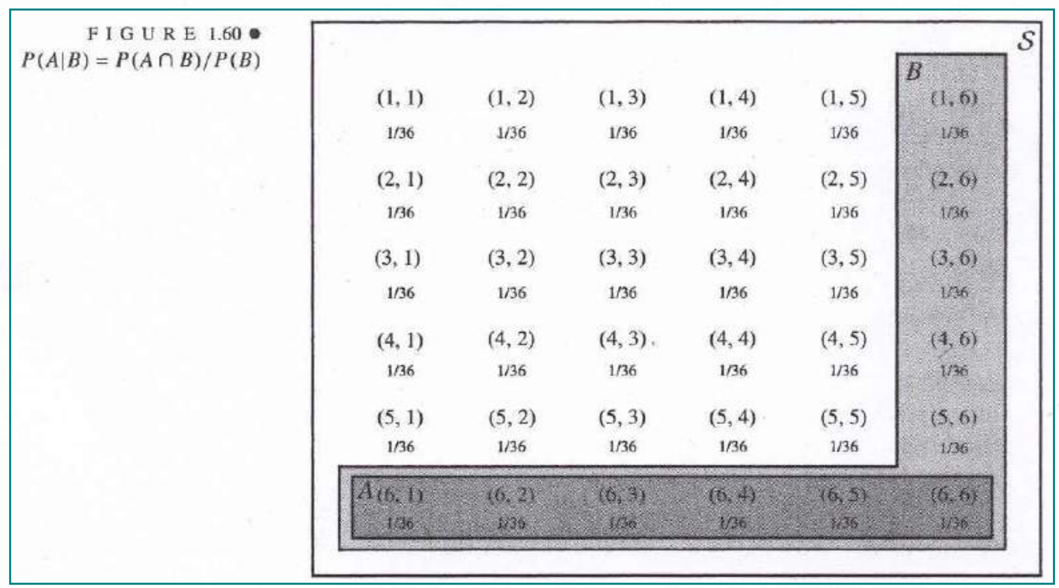
    B = { at least one 6 is obtained on the two dice }

$$P(A)=\frac{6}{36}=\frac{1}{6}\ \text{and}\ P(B)=\frac{11}{36}$$

$$P(A\,|\,B)=\frac{P(A\cap B)}{P(B)}$$

$$=\frac{P(A)}{P(B)}$$

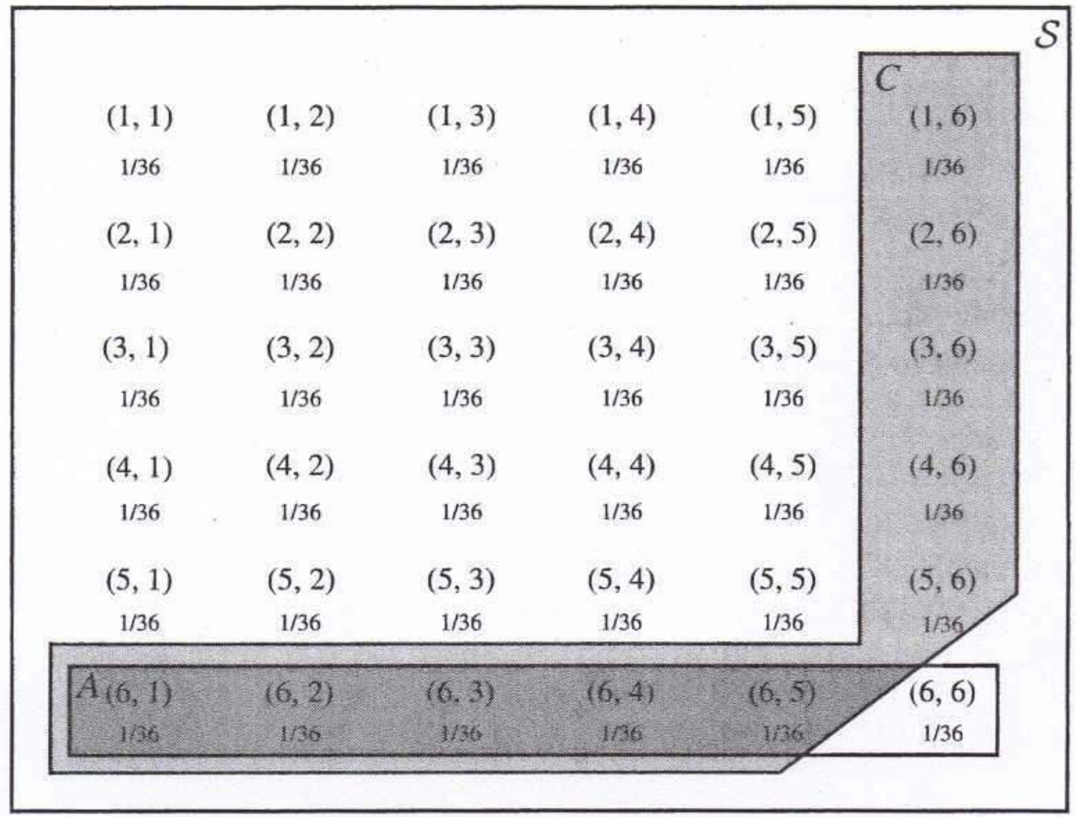
$$=\frac{1/6}{11/36}=\frac{6}{11}$$



1.4.2 Examples of Conditional Probabilities(3/3)

C = { exactly one 6 has been scored }

FIGURE 1.61 ●
$P(A|C) = P(A \cap C)/P(C)$



$$P(A|C) = \frac{P(A \cap C)}{P(C)}$$

$$= \frac{5/36}{10/36} = \frac{1}{2}$$

1.5 Probabilities of Event Intersections

1.5.1 General Multiplication Law

- $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

  $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$

  $\Rightarrow P(A \cap B) = P(B)P(A \mid B) = P(A)P(B \mid A)$

- $P(C \mid A \cap B) = \dfrac{P(A \cap B \cap C)}{P(A \cap B)}$

  $\Rightarrow P(A \cap B \cap C) = P(A \cap B)P(C \mid A \cap B)) = P(A)P(B \mid A)P(C \mid A \cap B)$

- Probabilities of Event Intersections

The probability of the **intersection of a series of events** $A_1, A_2, \ldots, A_n$ can be calculated from the expression

$$P(A_1 \cap \cdots \cap A_n) = P(A_1)P(A_2 \mid A_1) \cdots P(A_n \mid A_1 \cap \cdots \cap A_{n-1})$$

1.5.2 Independent Events

$$\square P(B|A)=P(B)$$

$$\Rightarrow P(A\cap B)=P(A)P(B|A)=P(A)P(B)$$

$$\text{and } P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{P(A)P(B)}{P(B)}=P(A)$$

- **Independent Events**

  Two events A and B are said to be **independent** events if

  one of the following holds:

  $$P(A|B)=P(A), P(B|A)=P(B), \text{ and } P(A\cap B)=P(A)P(B)$$

  Any one of these conditions implies the other two.

- The interpretation of two events being independent is that knowledge about one event does not affect the probability of the other event.
- **Intersections of Independent Events**

  The probability of the intersection of a series of **independent events** $A_1, A_2, ..., A_n$ is simply given by

$$P(A_1 \cap \cdots \cap A_n) = P(A_1)P(A_2)\cdots P(A_n)$$

1.5.3 Examples and Probability Trees(1/3)

- Example 7: **Car Warranties**

  A company sells a certain type of car, which it assembles in one of four possible locations. Plant I supplies 20%; plant II, 24%; plant III, 25%; and plant IV, 31%. A customer buying a car does not know where the car has been assembled, and so the probabilities of a purchased car being from each of the four plants can be thought of as being 0.20, 0.24, 0.25, and 0.31.

  Each new car sold carries a 1-year bumper-to-bumper warranty.

$$P(\,claim \mid plant\ I\,) = 0.05,\quad P(\,claim \mid plant\ II\,) = 0.11$$
$$P(\,claim \mid plant\ III\,) = 0.03,\quad P(\,claim \mid plant\ IV\,) = 0.08$$

  For example, a car assembled in plant I has a probability of 0.05 of receiving a claim on its warranty.

  Notice that claims are clearly not independent of assembly location because these four conditional probabilities are unequal.

1.5.3 Examples and Probability Trees(2/3)

$P(\text{claim}) = P(\text{plant I, claim}) + P(\text{plant II, claim})$
$\quad + P(\text{plant III, claim}) + P(\text{plant IV, claim})$
$\quad = 0.0687$

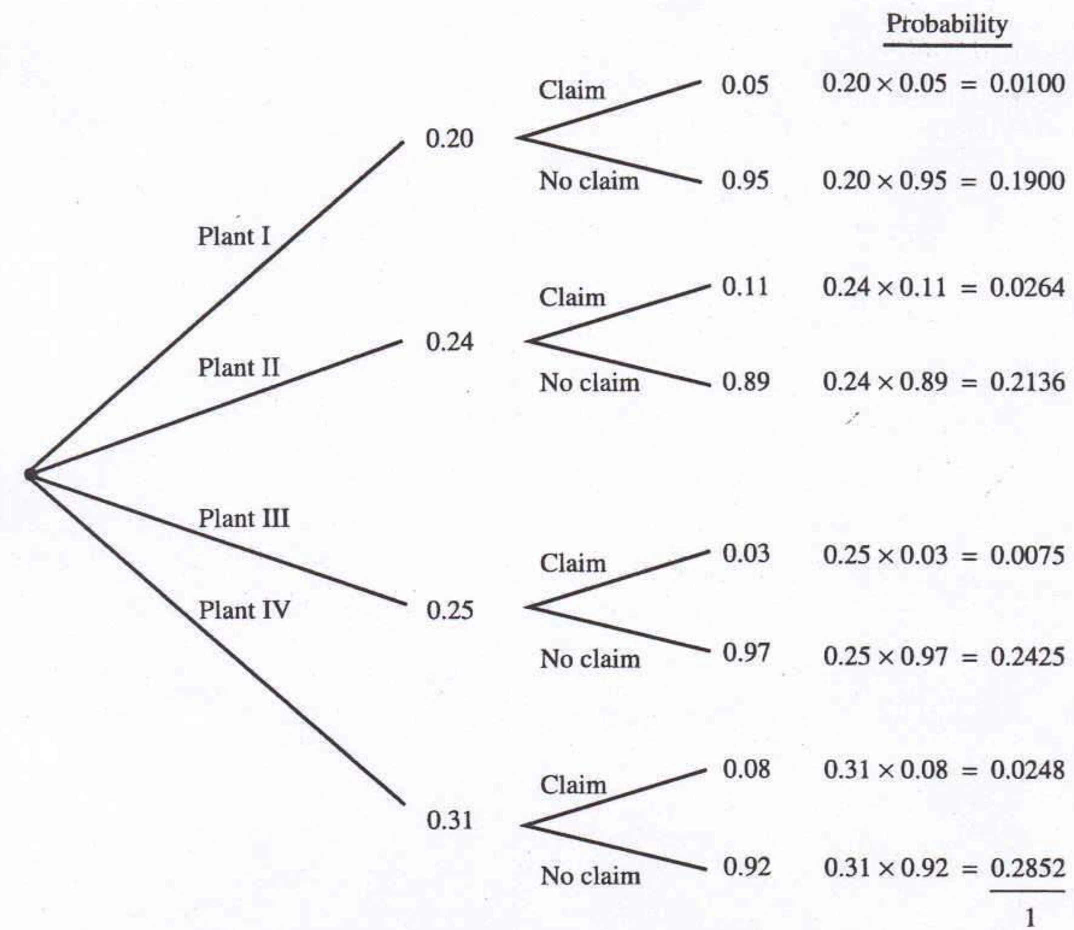


FIGURE 1.65 ● Probability tree for car warranties example

1.5.3 Examples and Probability Trees(3/3)

- GAMES OF CHANCE
  - A fair die

    even = { 2,4,6 } and high score = { 4,5,6 }

    Intuitively, these two events are not independent.

$$P(even) = \frac{1}{2} \quad \text{and} \quad P(even \mid high\ score) = \frac{2}{3}$$

  - A red die and a blue die are rolled.

    A = { the red die has an even score }

    B = { the blue die has an even score }

$$P(A \cap B) = P(A)P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

1.6 Posterior Probabilities
1.6.1 Law of Total Probability(1/3)

$S = \{1, 2, 3, 4, 5, 6\}$

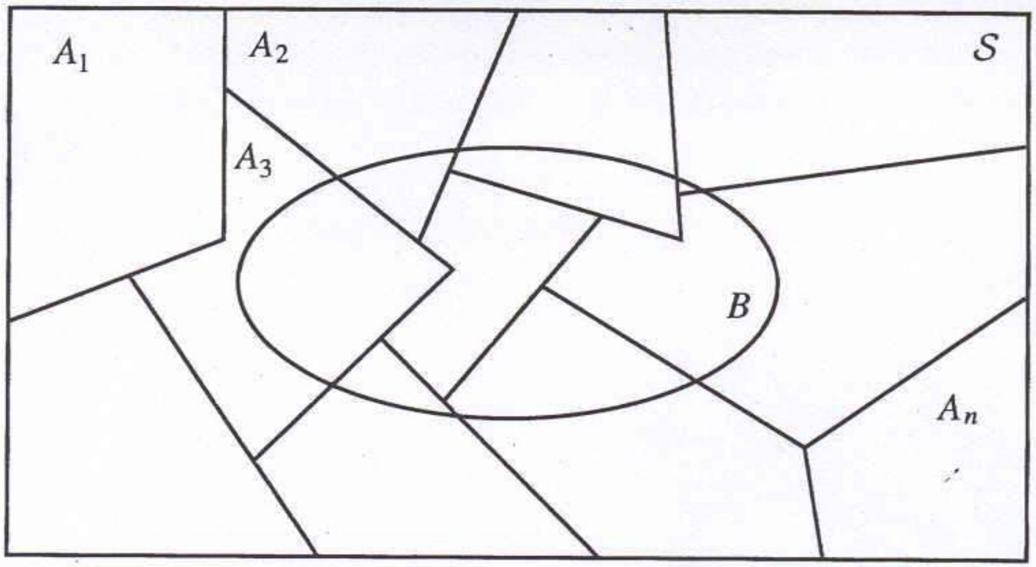
$S = A_1 \cup \cdots \cup A_n$ and A_i : mutually exclusive

$\Rightarrow B = (A_1 \cap B) \cup \cdots \cup (A_n \cap B)$ and $(A_i \cap B)$: mutually exclusive

$\Rightarrow P(B) = P(A_1 \cap B) + \cdots + P(A_n \cap B)$

$= P(A_1)P(B | A_1) + \cdots + P(A_n)P(B | A_n)$

FIGURE 1.67 ●
A partition $A_1, \ldots, A_n$ and an event B



1.6.1 Law of Total Probability(2/3)

- Law of Total Probability

If $A_1, A_2, ..., A_n$ is a partition of a sample space, then the probability of an event *B* can be obtained from the probabilities $P(A_i)$ and $P(B|A_i)$ using the formula

$$P(B) = P(A_1)P(B|A_1) + \cdots + P(A_n)P(B|A_n)$$

1.6.1 Law of Total Probability(3/3)

- Example 7 **Car Warranties**

  If $A_1, A_2, A_3, \text{ and } A_4$ are, respectively, the events that a car is assembled in plants I, II, III, and IV, then they provide a partition of the sample space, and the probabilities $P(A_i)$ are the supply proportions of the four plants.

  B = { a claim is made }

  = the claim rates for the four individual plants

$$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3) + P(A_4)P(B|A_4)$$

$$= (0.20 \times 0.05) + (0.24 \times 0.11) + (0.25 \times 0.03) + (0.31 \times 0.08)$$

$$= 0.0687$$

1.6.2 Calculation of Posterior Probabilities

- $P(A_i)$ and $P(B \mid A_i) \Rightarrow P(A_i \mid B) = ?$
- $P(A_1), \cdots, P(A_n)$: the prior probabilities
- $P(A_1 \mid B), \cdots, P(A_n \mid B)$: the posterior probabilities

$$\Rightarrow P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B \mid A_i)}{P(B)} = \frac{P(A_i)P(B \mid A_i)}{\sum_{j=1}^{n} P(A_j)P(B \mid A_j)}$$

- Bayes' Theorem

  If $A_1, A_2, ..., A_n$ is a partition of a sample space, then the **posterior probabilities** of the event A_i conditional on an event *B* can be obtained from the probabilities $P(A_i)$ and $P(B \mid A_i)$ using the formula

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{\sum_{j=1}^{n} P(A_j)P(B \mid A_j)}$$

1.6.3 Examples of Posterior Probabilities(1/2)

- Example 7 **Car Warranties**
  - The prior probabilities

$$P(plant\ I) = 0.20, \quad P(plant\ II) = 0.24$$

$$P(plant\ III) = 0.25, \quad P(plant\ IV) = 0.31$$

  - If a claim is made on the warranty of the car, how does this change these probabilities?

$$P(plant\ I\,|\,claim) = \frac{P(plant\ I)P(claim\,|\,plant\ I)}{P(claim)} = \frac{0.20 \times 0.05}{0.0687} = 0.146$$

$$P(plant\ II\,|\,claim) = \frac{P(plant\ II)P(claim\,|\,plant\ II)}{P(claim)} = \frac{0.24 \times 0.11}{0.0687} = 0.384$$

$$P(plant\ III\,|\,claim) = \frac{P(plant\ III)P(claim\,|\,plant\ III)}{P(claim)} = \frac{0.25 \times 0.03}{0.0687} = 0.109$$

$$P(plant\ IV\,|\,claim) = \frac{P(plant\ IV)P(claim\,|\,plant\ IV)}{P(claim)} = \frac{0.31 \times 0.08}{0.0687} = 0.361$$

1.6.3 Examples of Posterior Probabilities(2/2)

– *No* claim is made on the warranty

$$P(plant\ I \mid no\ claim) = \frac{P(plant\ I)P(no\ claim \mid plant\ I)}{P(no\ claim)}$$

$$= \frac{0.20 \times 0.95}{0.9313} = 0.204$$

$$P(plant\ II \mid no\ claim) = \frac{P(plant\ II)P(no\ claim \mid plant\ II)}{P(no\ claim)}$$

$$= \frac{0.24 \times 0.89}{0.9313} = 0.229$$

$$P(plant\ III \mid no\ claim) = \frac{P(plant\ III)P(no\ claim \mid plant\ III)}{P(no\ claim)}$$

$$= \frac{0.25 \times 0.97}{0.9313} = 0.261$$

$$P(plant\ IV \mid no\ claim) = \frac{P(plant\ IV)P(no\ claim \mid plant\ IV)}{P(no\ claim)}$$

$$= \frac{0.31 \times 0.92}{0.9313} = 0.306$$